Tata lectures on the curve

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The curve: * F characteristic p perfectoid field

$\rightarrow$ F perfect + Complete w.r.t. $|\cdot| : F \longrightarrow \mathbb{R}_+$
non trivial

Ex: * $\widehat{\overline{\mathbb{F}_p((T))}} \simeq \mathbb{C}_p^{\flat}$

* $\widehat{\mathbb{F}_p((T^{1/p^\infty}))} := \widehat{\bigcup_n \mathbb{F}_p((T^{1/p^n}))} \simeq \mathcal{O}_p^{cyc,\flat}$

Consequence of Fontaine-Wintenberger

* E local field $\nearrow [E : \mathbb{Q}_p] < +\infty$ $\quad \mathcal{O}_E/\pi = \mathbb{F}_q$
$\searrow \mathbb{F}_q((\pi))$

* Suppose $F|\mathbb{F}_q$.

$\rightsquigarrow X_{F,E}$ the curve attached to F and E

2 incarnations: * X = Dedekind scheme

$$\begin{array}{c} X \\ \downarrow \text{not of finite type} \\ \mathrm{Spec}(E) \end{array}$$

$\rightarrow$ analogy of a proper smooth curve

* X = adic space → fancy generalization of Tate rigid analytic spaces

$\downarrow$ not topologicaly of finite type

Spa E

analog of a (p-adic) Compact Riemann surface.

The space $Y_{F,E}$:

* Case $E = \mathbb{F}_q((\pi))$:

$$Y = \mathbb{D}_F^* = \{0 < |\pi| < 1\} \subset \mathbb{A}^1_F$$

$\downarrow$ Stein rigid analytic space $/F$

Spa(F)

$\Downarrow$

Y is completely determined by the Frechet algebra $\mathcal{O}(Y)$ + topology of uniform C.V. on q.c. subsets.

$$\mathcal{O}(Y) = \left\{ \sum_{n \in \mathbb{Z}} x_n \pi^n \;/\; x_n \in F \text{ and } \forall \rho \in]0,1[\ \lim_{|n| \to +\infty} |x_n| \rho^n = 0 \right\}$$

Frechet structure defined by Gauss norms $(|\cdot|_\rho)_{\rho \in]0,1[}$

$$\left|\sum_n x_n \pi^n\right|_\rho = \sup_n |x_n| \rho^n$$

classic Tate rigid space (loc. top. of finite type)

not locally topologicaly of finite type

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ \mathrm{Spa}\, F & & \mathbb{D}^*_{\mathbb{F}_q} = \mathrm{Spa}\left(\mathbb{F}_q((\pi))\right) = \mathrm{Spa}\, E \end{tikzcd}$$

trivial valuation

this is the structural morphism we are interested in

$\mathbb{F}_q((\pi)) \subset \mathcal{O}(Y)$

* Case $E|\mathbb{Q}_p$:

$$W_{\mathcal{O}_E}(\mathcal{O}_F) = \Big\{ \overbrace{\sum_{n\geq 0} [x_n]}^{\text{holomorphic function of the variable } p.} \pi^n \;/\; x_n \in \mathcal{O}_F \Big\}$$

ramified Witt vectors

Fontaine's $A_{\inf}$ when $E = \mathbb{Q}_p$

$$W_{\mathcal{O}_E}(\mathcal{O}_F)\left[\tfrac{1}{\pi}, \tfrac{1}{[\varpi_F]}\right] = \Big\{ \sum_{n >> -\infty} [x_n]\pi^n \;/\; \sup_n |x_n| < +\infty \Big\}$$

$$\subset W_{\mathcal{O}_E}(F)\left[\tfrac{1}{\pi}\right]$$

$\overset{\cdot\cdot}{\|}$

B^b

→ morally holomorphic functions on Y

⌊ Can give a precise meaning to this

that are meromorphic along the divisors (π) and $([\varpi])$

where $\varpi \in F$ is a pseudo-uniformizer i.e. $0 < |\varpi| < 1$.

Now define for $\rho \in]0,1[$ and $f = \sum_{n \gg 0} [x_n] \pi^n \in B^b$

$$\|f\|_\rho = \sup_n |x_n| \rho^n$$

<u>Def</u>: $B := \mathcal{O}(Y) =$ Frechet algebra obtained by completing B^b w.r.t. $(\|\cdot\|_\rho)_{\rho \in]0,1[}$.

⚠ : Contrary to the case $E = \mathbb{F}_q((\pi))$ elements of B can not be written as $\sum_{n \in \mathbb{Z}} [x_n] \pi^n$ with $(x_n)_n \in F^{\mathbb{Z}}$ and $\forall \rho \in]0,1[$ $\lim_{|n| \to +\infty} |x_n| \rho^n = 0$.

→ no Laurent expansion around $\pi = 0$ in general

In fact:

Th: One can define $Y := \mathrm{Spa}\left(W_{\mathcal{O}_E}(\mathcal{O}_F)\right) \setminus V(\pi[\varpi])$ as an adic space $/E$.

Content = Huber's presheaf is a sheaf

generalization of Tate's acyclicity lemma in a context not topologically of finite type.

$\to$ Y is preperfectoid $/E$ i.e. $Y \hat{\otimes}_E \mathbb{C}_p$ is perfectoid $/\mathbb{C}_p$ $\Rightarrow$ result

Ex: $E = \mathbb{F}_q((\pi))$ $\quad \mathbb{D}_F^{*} \hat{\otimes}_{\mathbb{F}_q((\pi))} \mathbb{F}_q((\pi^{1/p^\infty})) = \underbrace{\mathbb{D}_F^{*,1/p^\infty}}_{\varprojlim_{\mathrm{Frob}} \mathbb{D}_F^{*}}$ = perfectoid

But Y Stein $\Rightarrow$ $\mathcal{O}(Y)$ determines Y and you don't care (up to now...).

The Curve:

$Y \circlearrowleft \varphi$ Frobenius totally discontinuous action without fixed points.

induced by the Frob. of F

* $E = \mathbb{F}_q((\pi))$

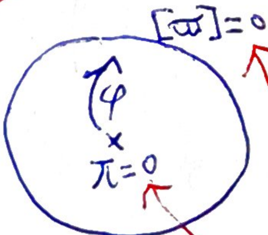
$\mathbb{D}^*_F \circlearrowleft \varphi$ given by

$$\varphi\left(\sum_n x_n \pi^n\right) = \sum_n x_n^q \pi^n$$

If we fix $|\cdot|$ on F, φ(annulus wt. radius ρ) , $\rho \in]0,1[$

$\parallel$

annulus wt. radius $\rho^{1/q}$



φ = arithmetic Frobenius, dilates.

two boundary divisors of Y

implies tot. disc. wt. fixed points

[Def: $X^{ad} = Y/\varphi^{\mathbb{Z}}$ = quasicompact E-adic space]

not topologically of finite type

$$\begin{tikzcd} & \mathbb{D}^*_F/\varphi^{\mathbb{Z}} \arrow[dl, "\times"'] \arrow[dr] & \\ \mathrm{Spa}(F) & & \mathrm{Spa}(E) \end{tikzcd}$$

not φ invariant

φ-invariant

$*$ $E|\mathbb{Q}_p$: $\varphi\left(\sum_n [x_n]\pi^n\right) = \sum_n [x_n^q]\pi^n$

usual Frob. of the Witt vectors. Extends by continuity to $B := \mathcal{O}(Y)$.

$X^{ad} := Y/\varphi^{\mathbb{Z}}$ = E-adic space.

The schematical curve

There is a line bundle $\mathcal{O}(1)$ on X^{ad}

géo. realization = $Y \times_{\varphi^{\mathbb{Z}}} \mathbb{A}^1$ where $\mathbb{A}^1 \ni \varphi = \underbrace{\times\pi^{-1}}_{\text{automorphy factor}}$

$$H^0(X^{ad}, \mathcal{O}(d)) = B^{\varphi=\pi^d}$$

$\{f \in \mathcal{O}(Y) \mid \varphi(f) = \pi^d f\}$

Declare $\mathcal{O}(1)$ ample :

$$\boxed{\underline{\text{Def}}: \ X = \mathrm{Proj}\left(\bigoplus_{d\geq 0} \mathcal{O}(Y)^{\varphi=\pi^d}\right)}$$

Zeros of holomorphic functions on Y and structure of X.

$$\mathcal{O}(Y)^+ = \text{fct. on } Y \text{ bounded by } 1 = \begin{cases} \mathcal{O}_F[[\pi]] \\ W_{\mathcal{O}_E}(\mathcal{O}_F) \end{cases}$$

$\underline{\text{Def}}$. $f = \sum_{n\geq 0}' [x_n]\pi^n \in \mathcal{O}(Y)^+$ is primitive of degree $d \geq 1$ if $x_0 \neq 0$, $x_0, \dots, x_{d-1} \in \mathfrak{m}_F$ and $x_d \in \mathcal{O}_F^\times$

$\rightarrow$ notion comes from Weierstrass factorization theory

prim. deg. d $\times$ prim. deg. d' = prim. deg. $d+d'$.

$\rightsquigarrow$ good notion of primitive irreducible.

Th: If F is alg. closed then $\forall f$ primitive of deg. d

$$f = \underbrace{u}_{\text{unit in } \mathcal{O}(Y)^{+}} \times (\pi - [a_1]) \times \dots \times (\pi - [a_d])$$

$0 < |a_i| < 1$, a_i not uniquely determined if $E|\mathbb{Q}_p$

$\Rightarrow$ any primitive irreducible is of degree 1.

Th: $\forall \xi$ primitive irreducible if $K = \mathcal{O}(Y)/\xi$ then $K|E$ is a perfectoid field equipped wt. an embedding

$$F \hookrightarrow K^{\flat} \quad \text{s.t. } [K^{\flat} : F] = \deg(\xi)$$

$$x \longmapsto \left([x^{p^{-n}}] \bmod \xi\right)_{n \geq 0}$$

Def: $|Y|^{cl} = \{$ zeros of irreducible primitive elements $\}$

classical Tate rigid points

Th: $|Y|^{cl,\, deg=1} \xrightarrow{\sim} \{\text{untilts of } F \text{ over } E\} = $ untilts of F if $E = \mathbb{Q}_p$

$$\{(K,\iota) \mid K|E \text{ perfectoid}, \ \iota : F \xrightarrow{\sim} K^\flat\}/\sim$$

Th: $\mathcal{C} \subset Y$ quasi-compact annulus

$\hookrightarrow$ of the form $\{|[\varpi]|^\alpha \le |\pi| \le |[\varpi]|^\beta\}$ (variable)

Then the Banach algebra $\mathcal{O}(\mathcal{C})$ is a P.I.D. with $\mathrm{Spm}(\mathcal{O}(\mathcal{C})) = |\mathcal{C}| \cap |Y|^{cl}$.

+ plenty of factorization results involving Newton polygons and $|Y|^{cl}$.

$\Rightarrow$ X^{ad} has an affinoid cover by Spa(P.I.D. Banach algebras)

$\longrightarrow$ this is a curve

Th: X is a noetherian regular scheme of dimension 1 (i.e. $\bigcup_{\text{finite}} \mathrm{Spec}$ (Dedekind rings)) s.t.

(1) $|Y|^{cl}/\varphi^{\mathbb{Z}} \xrightarrow{\sim} \underbrace{|X|}_{\text{closed points}}$ + morphism of ringed spaces $X^{ad} \longrightarrow X$

(2) If $\underset{|Y|^{cl}}{\overset{\cap}{y}} \longmapsto \underset{|X|}{\overset{\cap}{x}}$ then $k(y) = k(x)$ | E is a perfectoid field

s.t. $\underbrace{[k(x)^\flat : F]}_{\substack{\| \\ \deg(x)}} < +\infty$

Fontaine's θ identi with quotient to residue fields

and $\underbrace{\widehat{\mathcal{O}_{X,x}}}_{\text{D.V.R.}} \xrightarrow{\sim} \widehat{\mathcal{O}_{Y,y}} = B^+_{dR}(k(x)) \overset{\theta}{\twoheadrightarrow} k(x)$

(3) $\forall f \in \underbrace{E(X)}_{\mathcal{O}_{X,\eta}}^\times \quad \deg(\operatorname{div} f) = 0$ i.e. "X is complete"

$\hookrightarrow \deg(x) = [k(x)^\flat : F]$. ← 1 if F is alg. closed

(4) $\forall t \in \underbrace{H^0(X, \mathcal{O}(1))}_{B^{\varphi=\pi}} \setminus \{0\}$, $\underbrace{V^+(t)}_{\text{vanishing locus of hyperplane section}} = \{\infty\}$

If F is alg. closed, $b(\infty) = C$ w.t. $C^b = F$

$$H^0(X\setminus\{\infty\}, \mathcal{O}_X) = B_e = B\left[\tfrac{1}{t}\right]^{\varphi = Id} = B_{cris}(C)^{\varphi = Id}$$

$$X \setminus \{\infty\} = \mathrm{Spec}(B_e)$$

B_e = P.I.D.

$(B_e, -\mathrm{ord}_\infty)$ non euclidean, almost euclidean

valuation on $\hat{\mathcal{O}}_{X,\infty} = B^+_{dR}$

$\forall x, y \ \exists a, b$

$x = ay + b$, $\deg(b) \leq \deg($